Green's Theorem

Idea: Relate line integrals to double integrals.

Prop (Green's Theorem): Let C be a positively oriented, piecewise smooth, simple closed curve in $\mathbb{R}^2$, and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region containing D, then



Boundary of D $\longrightarrow$
$$\int_C P\,dx + Q\,dy \overset{*}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

NB: We often write ∂D instead of C to denote the boundary of D.

Ex: Compute $\int_C y\,dx - x\,dy$ with C the circle of radius 4 centered at the origin directly and then via Green's Theorem.

Sol (Direct): $\vec{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle$ on $0 \le t \le 2\pi$.

So $\vec{r}\,'(t) = \langle -4\sin(t), 4\cos(t)\rangle$.

$$\therefore \int_C y\,dx - x\,dy = \int_{t=0}^{2\pi} \left(4\sin(t)\cdot -4\sin(t) - 4\cos(t)\cdot 4\cos(t)\right) dt$$

$$= -16\int_{t=0}^{2\pi} dt = -16t\Big|_{t=0}^{2\pi} = -32\pi \quad \boxed{//}$$

Sol (Green's Theorem): $\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$$\therefore \int_C y\,dx + (-x)\,dy = \iint_D \left(\frac{\partial}{\partial x}[-x] - \frac{\partial}{\partial y}[y]\right) dA = -2\iint_D dA$$

$$= -2\,\text{Area}(D) = -2\cdot\pi\cdot 4^2 = -32\pi \quad \boxed{//}$$

NB: In general, we may need to solve $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ via some other method.

Ex: Compute $\int_C \left(3y - e^{\sin(x)}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy$ for C the circle with equation $x^2+y^2=9$.

Sol: Using Green's theorem:

$$\int_{\partial D} \left(3y - e^{\sin(x)}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy$$

$$= \iint_D \left(\frac{\partial}{\partial x}\left[7x + \sqrt{y^4+1}\right] - \frac{\partial}{\partial y}\left[3y - e^{\sin(x)}\right] \right) dA$$

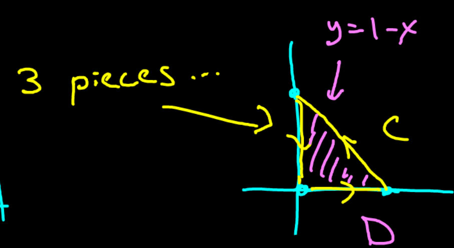
$$= \iint_D (7-3)\,dA = 4\iint_D dA = 4\,\text{area}(D)$$

Now D is a circle of radius 3, so $\text{area}(D) = \pi \cdot 3^2 = 9\pi$.

Hence $\int_C \left(3y - e^{\sin(x)}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy = 36\pi$ //

Ex: Compute $\int_C x^4\,dx + xy\,dy$ for C the positively oriented curve along the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$.

Sol: Using Green's Theorem:



$$\int_C x^4\,dx + xy\,dy = \iint_D \left(\frac{\partial}{\partial x}[xy] - \frac{\partial}{\partial y}[x^4] \right) dA$$

$D = \left\{ (x,y) : \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{matrix} \right\}$

$$= \iint_D y\,dA = \int_{x=0}^{1} \int_{y=0}^{1-x} y\,dy\,dx$$

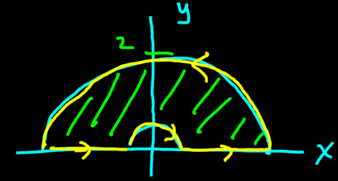
$$= \int_{x=0}^{1} \frac{1}{2}\left[y^2\right]_{y=0}^{1-x} dx = \frac{1}{2}\int_{x=0}^{1} (1-x)^2\,dx$$

$\left(\begin{matrix} u = 1-x & u(1) = 0 \\ du = -dx & u(0) = 1 \end{matrix} \right.$

$$= \frac{1}{2}\int_{u=1}^{0} -u^2\,du = -\frac{1}{2}\cdot\frac{1}{3}u^3\Big|_{u=1}^{0} = -\frac{1}{6}(0-1) = \frac{1}{6}$$ //

Ex: Compute $\int_{\partial R} y^2\,dx + 3xy\,dy$ for $R = \{(x,y) : 1 \le x^2+y^2 \le 4,\ y \ge 0\}$.

Sol: Use Green's Theorem:



$$\int_{\partial R} y^2 dx + 3xy\,dy = \iint_R \left(\frac{\partial}{\partial x}[3xy] - \frac{\partial}{\partial y}[y^2]\right) dA$$

$$= \iint_R (3y - 2y)\,dA = \iint_R y\,dA$$

$R_{r,\theta} = \{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

$dA_{xy} = r\,dA_{r\theta}$

Change to polar coordinates! $(x,y) = (r\cos(\theta), r\sin(\theta))$

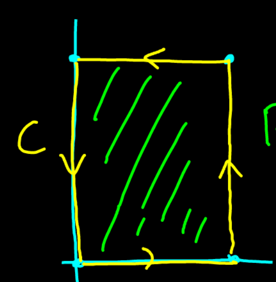
$$= \iint_{R_{r,\theta}} r\sin(\theta)\,r\,dA_{r\theta} = \int_{r=1}^{2}\int_{\theta=0}^{\pi} r^2 \sin(\theta)\,d\theta\,dr$$

$$= \int_{r=1}^{2} r^2 \left[-\cos(\theta)\right]_{\theta=0}^{\pi} dr = (\cos(0) - \cos(\pi)) \int_{r=1}^{2} r^2\,dr$$

$$= 2 \cdot \frac{1}{3}\left[r^3\right]_{r=1}^{2} = \frac{2}{3}(8-1) = \frac{14}{3}$$

Ex: Compute $\int_C ye^x\,dx + 2e^x\,dy$ for C the rectangle with vertices $(0,0)$, $(3,0)$, $(3,4)$, and $(0,4)$.

Sol: Using Green's Theorem:



$$\int_C ye^x dx + 2e^x dy = \iint_R \left(\frac{\partial}{\partial x}[2e^x] - \frac{\partial}{\partial y}[ye^x]\right) dA$$

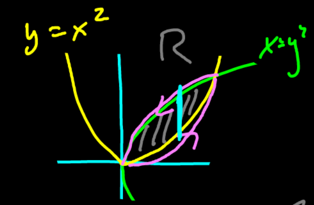
$$= \iint_R 2e^x - e^x\,dA = \int_{x=0}^{3}\int_{y=0}^{4} e^x\,dy\,dx$$

$R = \{(x,y) : 0 \le x \le 3,\ 0 \le y \le 4\}$

$$= \int_{x=0}^{3} \left[ye^x\right]_{y=0}^{4} dx = \int_{x=0}^{3} 4e^x\,dx = 4e^x\Big|_{x=0}^{3} = 4(e^3 - e^0) = 4e^3 - 4$$

Ex: Compute $\int_C (y + e^{\sqrt{x}})\,dx + (2x + \cos(y^2))\,dy$ where C is the boundary of the region enclosed by $x = y^2$ and $y = x^2$.

Sol: Using Green's Theorem:



$$R = \left\{ (x,y) : \begin{array}{l} 0 \le x \le 1, \\ x^2 \le y \le \sqrt{x} \end{array} \right\}$$

$$\int_C (y + e^{\sqrt{x}})\,dx + (2x + \cos(y^2))\,dy$$

$$= \iint_R \left(\frac{\partial}{\partial x}\left[2x + \cos(y^2)\right] - \frac{\partial}{\partial y}\left[y + e^{\sqrt{x}}\right] \right) dA$$

$$= \iint_R (2 - 1)\,dA = \iint_R dA = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} dy\,dx$$

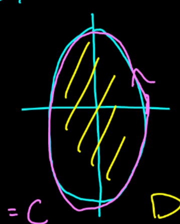
$$= \int_{x=0}^{1} \left[y \right]_{y=x^2}^{\sqrt{x}} dx = \int_{x=0}^{1} \left(x^{1/2} - x^2 \right) dx$$

$$= \left[\frac{2}{3} x^{3/2} - \frac{1}{3} x^3 \right]_{x=0}^{1} = \left(\frac{2}{3} - \frac{1}{3} \right) - (0 - 0) = \frac{1}{3}$$

$\boxed{//}$

Ex: Compute $\int_C y^4\,dx + 2xy^3\,dy$ for C the ellipse $x^2 + 2y^2 = 2$.

Sol: First solution: Parametrize the ellipse. $\left(\frac{x}{\sqrt{2}}\right)^2 + y^2 = 1$



$\vec{r}(\theta) = \langle \sqrt{2}\cos(\theta), \sin(\theta)\rangle$ on $0 \le \theta \le 2\pi$.

$\vec{r}\,'(\theta) = \langle -\sqrt{2}\sin(\theta), \cos(\theta)\rangle$

$$\therefore \int_{\partial D} y^4\,dx + 2xy^3\,dy = \int_{\theta=0}^{2\pi} \left(\sin^4(\theta)\cdot -\sqrt{2}\sin(\theta) + 2\sqrt{2}\cos(\theta)\cdot\sin^3(\theta)\cos(\theta)\right) d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(-\sqrt{2}\sin^5(\theta) + 2\sqrt{2}\cos^2(\theta)\sin^3(\theta)\right) d\theta$$

$$= \sqrt{2}\int_{\theta=0}^{2\pi} \left(2\left(1-\cos^2(\theta)\right)\cos^2(\theta) - \left(1-\cos^2(\theta)\right)^2\right)\sin(\theta)\,d\theta$$

$$= -\sqrt{2}\int_{u=1}^{1} \left(2(1-u^2)u^2 - (1-u^2)^2\right) du = 0 \qquad \left(\begin{array}{l} u = \cos(\theta) \\ du = -\sin(\theta)\,d\theta \end{array} \quad \begin{array}{l} u(2\pi) = 1 \\ u(0) = 1 \end{array}\right)$$

Second Solution: Using Green's theorem:

$D_{r,\theta} = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

where $\begin{cases} x = \sqrt{2}\,r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \sqrt{2}\cos(\theta) & -\sqrt{2}\,r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = \sqrt{2}\,r\left(\cos^2(\theta) + \sin^2(\theta)\right) = \sqrt{2}\,r.$$

$$\int_{\partial D} y^4\,dx + 2xy^3\,dy = \iint_D \left(\frac{\partial}{\partial x}\left[2xy^3\right] - \frac{\partial}{\partial y}\left[y^4\right]\right) dA$$

$$= \iint_D \left(2y^3 - 4y^3\right) dA = \iint_D -2y^3\,dA$$

$$= \iint_{D_{r,\theta}} -2r^3\sin^3(\theta)\cdot\sqrt{2}\,r\,dA_{r\theta} = 2\sqrt{2}\int_{r=0}^{1}\int_{\theta=0}^{2\pi} -r^4\sin^3(\theta)\,d\theta\,dr$$

$$= 2\sqrt{2}\int_{r=0}^{1} r^4 \int_{\theta=0}^{2\pi} \left(1-\cos^2(\theta)\right)\cdot -\sin(\theta)\,d\theta\,dr \qquad \left(\begin{array}{l} u = \cos(\theta) \\ du = -\sin(\theta)\,d\theta \end{array} \quad \begin{array}{l} u(2\pi) = 1 \\ u(0) = 1 \end{array}\right)$$

$$= 2\sqrt{2}\int_{r=0}^{1} r^4 \int_{u=1}^{1} (1-u^2)\,du\,dr = 2\sqrt{2}\int_{r=0}^{1} 0\,dr = 0 \quad \blacksquare$$

NB: $\text{Area}(R) = \iint_R dA = \int_{\partial R} P\,dx + Q\,dy$ for any functions P, Q satisfying $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ (because Green's Theorem)!

Ex: Compute the area of general ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol: Observe $P=0, Q=x$ satisfies $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$. ← Partial Differential Equation

$$\therefore \text{Area}(R) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial R} P\,dx + Q\,dy = \int_{\partial R} x\,dy$$

∂R has $\begin{cases} \vec{r}(t) = \langle a\cos(t), b\sin(t)\rangle \\ \vec{r}\,'(t) = \langle -a\sin(t), b\cos(t)\rangle \end{cases}$ on $0 \le t \le 2\pi$.

$$\therefore \text{Area}(R) = \int_{\partial R} x\,dy = \int_{t=0}^{2\pi} a\cos(t)\cdot b\cos(t)\,dt$$

$$= \frac{ab}{2}\int_{t=0}^{2\pi} (1+\cos(2t))\,dt = \frac{ab}{2}\left[t + \tfrac{1}{2}\sin(2t)\right]_{t=0}^{2\pi}$$

$$= \frac{ab}{2}\left(\left(2\pi + \tfrac{1}{2}\sin(4\pi)\right) - \left(0 + \tfrac{1}{2}\sin(0)\right)\right) = \frac{ab}{2}\cdot 2\pi = ab\pi. \quad //$$

Sol 2: Again $\text{Area}(R) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial R} P\,dx + Q\,dy$

If $P = -\frac{1}{2}y$ and $Q = \frac{1}{2}x$, then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{2} - -\frac{1}{2} = 1$

(using $\vec{r}(t) = \langle a\cos(t), b\sin(t)\rangle$, $\vec{r}\,'(t) = \langle -a\sin(t), b\cos(t)\rangle$):

$$\text{Area}(R) = \int_{t=0}^{2\pi} \left(-\tfrac{1}{2}b\sin(t)\cdot -a\sin(t) + \tfrac{1}{2}a\cos(t)\cdot b\cos(t)\right) dt$$

$$= \frac{ab}{2}\int_{t=0}^{2\pi} \left(\sin^2(t) + \cos^2(t)\right) dt = \frac{ab}{2}\int_{t=0}^{2\pi} dt = \frac{ab}{2}\, t\Big|_{t=0}^{2\pi} = ab\pi \quad //$$

Point: $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ has lots of solutions!

If we're clever, the line integral is easy!

